Graphing

Other Handouts:
- Scientific notation • Units
- Significant Figures
- Ratio and Proportion
- Logarithms
- Review of Number

Graphing

Graphical representations provide a visual link between number, algebra and geometry in two dimensions. The two dimensions are represented by number lines placed at right angles. The **x axis** is the number line placed horizontally and the **y axis** is the number line placed vertically in the same plane. The two axes meet at the zero point on both number lines. This common zero point is called the **origin**. The axes form four **quadrants** (or quarters) of the plane, as shown in Figure 1.

An **ordered pair** is a pair of numbers placed within brackets and separated by a comma, to indicate a strict ordering. The first component of the ordered pair is a distance along the horizontal (x axis) number line. This is called the **x coordinate** of the ordered pair. The second component of the ordered pair is a distance along the vertical (y axis) number line. This is called the **y coordinate** of the ordered pair. Ordered pairs represent points in the **rectangular coordinate system** defined by the two axes.

The ordered pair (5, 2) has x coordinate of 5 (start at 5 along the x axis) and y coordinate of 2 (move from the 5 on the x axis up 2 units in the direction parallel to the y axis). This ordered pair is now represented by the point at the vertex of the rectangle formed by the x and y axes with lengths 5 and 2 respectively. The point would be referred to as the ordered pair (5, 2) and is labelled accordingly on the diagram. This process is called **plotting** the point (5, 2).

Any point on the plane of the coordinate axes can be represented by an ordered pair. To find the coordinates of a given point, move perpendicularly from the point to the x axis to find the x coordinate, and to the y axis to find the y coordinate. Again a rectangle is formed with the point at the leading vertex. The origin is the point (0, 0).

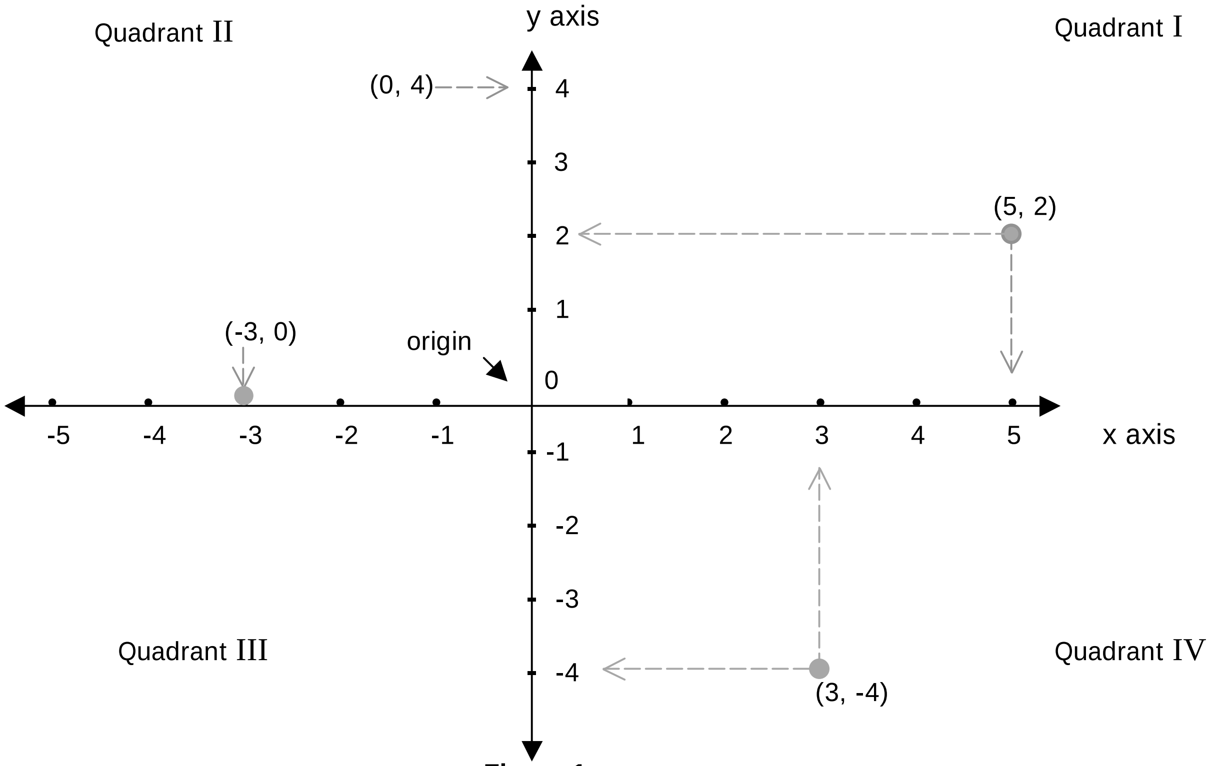


Figure 1

A **graph** is a set of ordered pairs plotted as points in the plane using the rectangular coordinate system. In Figure 1 we see that the point (5, 2) is in the 1st quadrant and (3, -4) is in the 4th quadrant. The point (-3, 0) lies on the x axis as its y coordinate is 0 (it does not rise off the x axis). The point (0, 4) lies on the y axis.

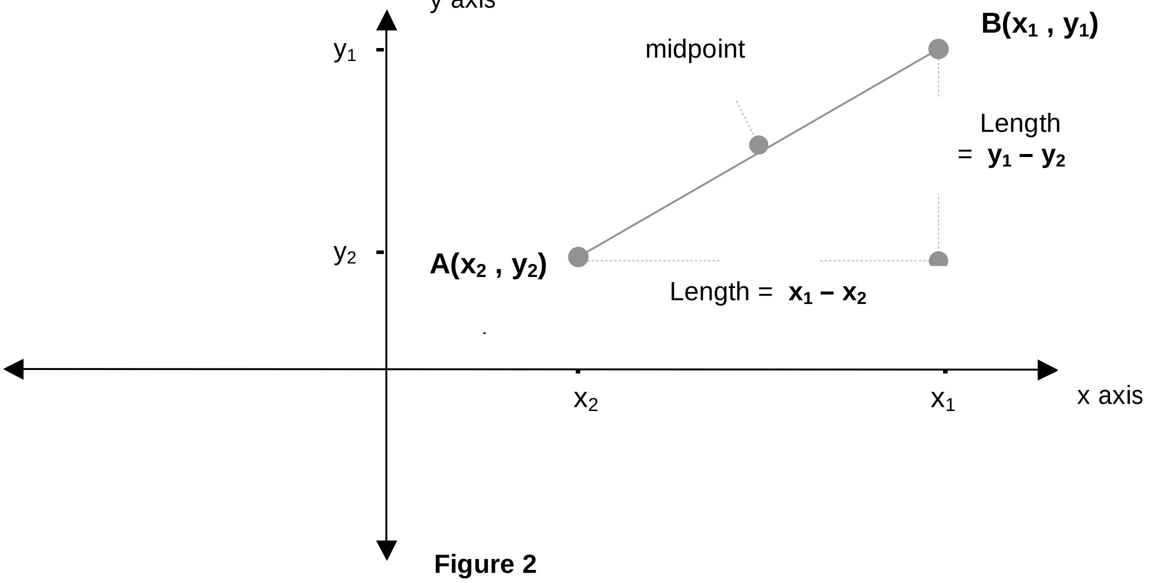


Figure 2

Distance and Midpoint Formulae

Given two points A and B with ordered pairs (x_1, y_1) and (x_2, y_2) we can draw the line segment between them as in Figure 2. The lengths between the points A and B along the x and y axes are the lengths of the sides of a right angle triangle as shown. Using Pythagoras' Theorem we can calculate the length of the line segment from A to B. We call this length the **distance** between (x_1, y_1) and (x_2, y_2).

$$\textbf{Distance} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Example The distance between the points A (2, 6) and B (0, -1) is (measured in the units used on the axes)

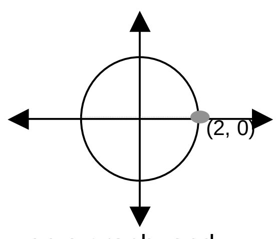
$$\sqrt{(2-0)^2 + (6-(-1))^2} = \sqrt{(2)^2 + (7)^2} = \sqrt{4 + 49} = \sqrt{53} \text{ units } (\approx 7.28 \text{ units})$$

The **midpoint** of the line segment from A to B is exactly half way between the points (x_1, y_1) and (x_2, y_2). It has coordinates which are the averages of the x and y coordinates of the original points.

$$\textbf{Midpoint} = \left(\frac{x_1 + x_2}{2} , \frac{y_1 + y_2}{2} \right)$$

Example The midpoint of the line segment connecting points A (2, 6) and B (0, -1) has x coordinate $\frac{2+0}{2} = 1$ and y coordinate $\frac{6+(-1)}{2} = {}^5/_2 = 2\frac{1}{2}$.

The midpoint is $(1, 2\frac{1}{2})$.



Equations of Circles

The points on a **circle** are all the same distance from the centre of the circle.
For example, a circle centred on the origin (0, 0) with radius 2 units is easy to draw on a graph, and passes through an infinite number of points in the plane of the x and y coordinate axes. If (x, y) was one of the points on the circle, its distance from the centre (0, 0) must be 2.

So $\sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2} = 2$ units.

Squaring both sides of this equation gives $x^2 + y^2 = 4$.

The equation $x^2 + y^2 = 4$ is called the **equation of the circle** as the x and y coordinates of every point on the circle must satisfy this equation. And conversely, if an ordered pair's x and y coordinates satisfy this equation, then that point must lie on this circle.

For example, the point (0, 2) lies on the circle with equation $x^2 + y^2 = 4$ (as $0^2 + 2^2 = 4$) and the point (1,3) does not lie on the circle (as $1^2 + 3^2 = 10$, not 4).

The general equation of a circle with centre **(h, k)** and radius **r** passing through the point **(x , y)** is

$$\mathbf{(x - h)^2 + (y - k)^2 = r^2}$$

For example, the equation of a circle with centre (1, 2) and radius 5 is given by

$$(x - 1)^2 + (y - 2)^2 = 5^2$$
$$\text{i.e.} \quad (x - 1)^2 + (y - 2)^2 = 25$$

The point (0, 0) does not lie on this circle as $(0 - 1)^2 + (0 - 2)^2 = 5$, not 25. The points (1, 7) and (-4, 2) lie on the circle as $(1 - 1)^2 + (7 - 2)^2 = 25$ and $(-4 - 1)^2 + (2 - 2)^2 = 25$

Practice Problems - Graphing

1. (a) Draw x and y axes and plot the points

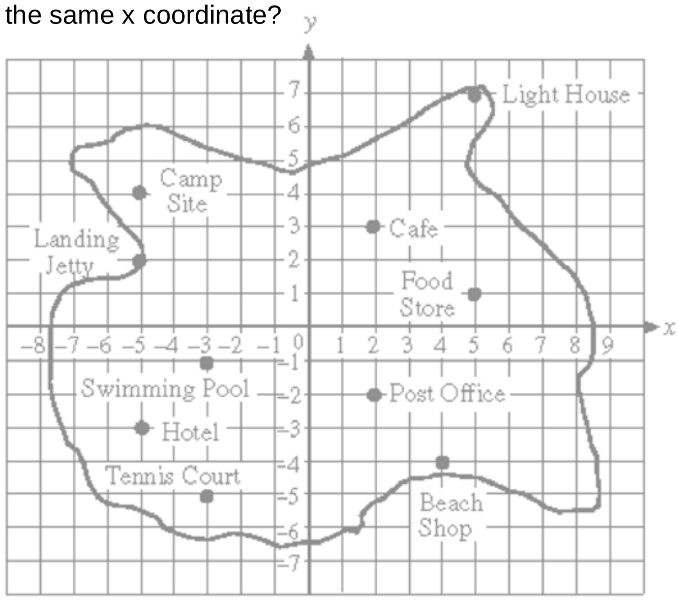
(4, 0)	(-1, 2)	(0, 4)	(3, -3)
(1, 2)	(1, -2)	(3, 3)	(0, -4)
(2, -1)	(-3, 3)	(-2, 1)	(-2, -1)
(-1, -2)	(-4, 0)	(2, 1)	(-3, -3)

Verify that the graph of these points lies on two concentric (same centre) rings.

2. (a) For the rings in Q1 what are the coordinates of the common centre?
(b) The inner ring is a circle. What is the radius of this circle? (c) Write the equation of this small circle.
(d) Do (0, √5) and (1½, 1½) lie on this circle? (e) Show the points on the outer ring do <u>not</u> lie on a circle.

3. The map below shows a small island. (a) What are the coordinates of the flowing places on the map?
(i) Hotel (ii) Beach Shop (iii) Lighthouse (iv) Landing Jetty?

(b) Can you identify 3 places on the map which have the same x coordinate?



(c) Lee walked from the point with coordinates (5, 1) to (-3, -1), then to (-5, -3) and finally through (2, -2) on to (-5, 4). What places did Lee pass through on the journey?

(d) If the scale on the graph is 1 unit = 100 m, how far away from the Café is the
(i) Post Office (ii) Lighthouse (iii) Swimming Pool?

(e) What are the coordinates of the position halfway between
(i) the Camp Site and the Landing Jetty
(ii) the Post Office and the Beach Shop
(iii) the Lighthouse and the Camp Site?

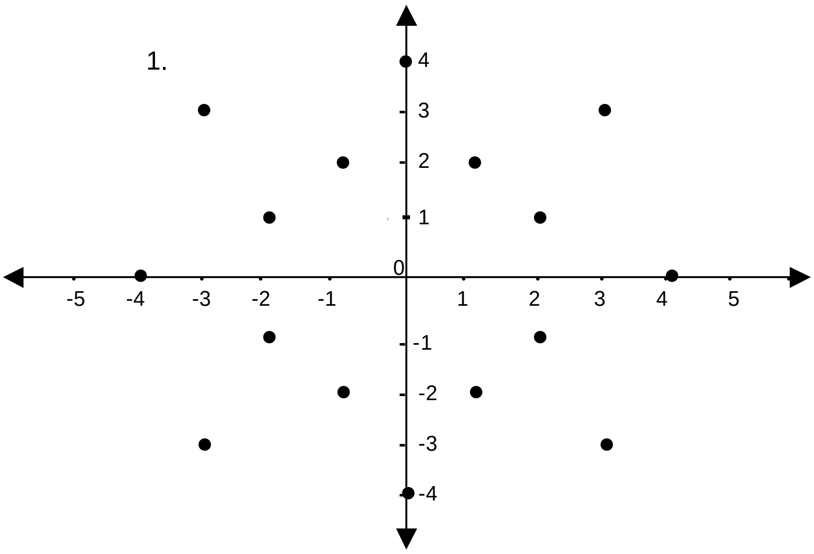
(f) If the x axis is pointing East and the y axis is pointing North, what would be the coordinates of the position
(i) 400 m North of the Swimming Pool (ii) 250 m South of the Café (iii) 500 m West of the Beach Shop
(iv) 200 m South and 300m East of the Food Store?

(g) (i) What would be your destination if you began at the Lighthouse and travelled 600 m South followed by 250 m East; then 450 m South and 1,350 m West; then 350 m North, followed by 300 m East; then finally 100 m South?
(ii) What distance would you have travelled if you had walked in a straight line directly from the Lighthouse to your destination?

(h) (i) Verify that the Swimming Pool and the Tennis Court are equidistant (same distance) from the Hotel.
(ii) What would be the equation of the circle, centred at the Hotel, which passes through both the Tennis Court and the Swimming Pool?
(iii) Find the coordinates of two more points lying on this circle.

Solutions to Practice Problems - Graphing

1.



2.(a) (0, 0) (b) √5 (c) $x^2 + y^2 = 5$
(d) (0, √5) only, since the distance from the centre (0, 0) to (1½, 1½) is not √5.
(e) Distance from (4,0) to the centre (0, 0) is 4 units. Distance from (3, 3) to (0, 0) is √18, not 4. So points do not lie on the same circle.

3.(a) (i) (-5, -3) (ii) (4, -4) (iii) (5, 7) (iv) (-5, 2)
(b) Campsite, Landing Jetty and Hotel
(c) Food Store, Swimming pool, Hotel, Post Office, Camp site.
(d) (i) Distance (2, 3) to (2, -2) = 5 units = 500m
(ii) Distance (2, 3) to (5, 7) = 5 units = 500m
(iii) Distance (2, 3) to (-3, -1) = √41 = 6.403 units ≈ 640.3 m
(e) (i) Midpoint of (-5, 4) and (-5, 2) = (-5, 3) (ii) Midpoint of (2, -2) and (4, -4) = (3, -3)
(iii) Midpoint of (5, 7) and (-5, 4) = (0, 5½)
(f) (i) (-3, 3) (ii) (2, ½) (iii) (-1, -4) (iv) (8, -1)
(g) (i) Swimming Pool (ii) Distance (5, 7) to (-3, -1) = √128 = 11.3137 units ≈ 1131.4 m
(h) (i) Distance (-3, -1) to (-5, -3) = √ 8 and Distance (-3, -5) to (-5, -3) = √ 8 (ii) $(x + 5)^2 + (y + 3)^2 = 8$
(iii) Any points satisfying this equation would lie on the circle but (-7, -1) and (-7 , -5) are clear from map.

Lines and their Slopes

A straight **line** is simple to draw by connecting two points in the rectangular coordinate system and continuing the graph in the same direction either side of the points. The resultant line makes a distinctive angle with the x axis that depends on the **slope** of the line. The slope is measured by calculating the distance along the x axis between **any** two points on the line (called the run), and the corresponding distance along the y axis (called the rise). The ratio of the rise to the run is the slope of the line.

$$\textbf{Slope} = \frac{\text{Rise}}{\text{Run}} = \frac{y_1 - y_2}{x_1 - x_2}$$

This slope is constant along the length of the line. It does not change and is characteristic of the given line.

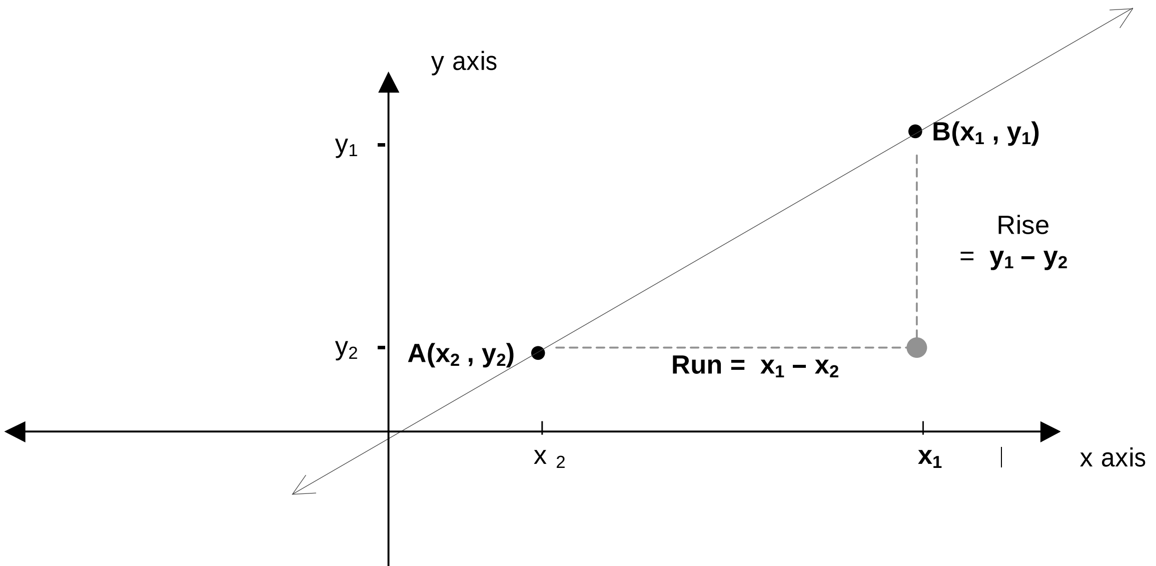
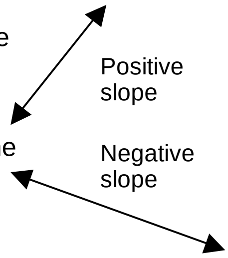


Figure 3

The slope of the line between two points may be interpreted as the **average rate of change** of the y coordinates with respect to the x coordinates for those points.

For a line graph with a **positive slope**, as the x coordinates of its ordered pairs become larger in size, so do the corresponding y coordinates. So an increase in x produces an increase in y for the points on the line.



For a line graph with a **negative slope**, as the x coordinates of its ordered pairs become larger in size, the corresponding y coordinates become smaller. So an increase in x produces a decrease in y for the points on the line.

Note that **parallel** lines would have the **same** slope, but pass through a different set of points. A line with **zero** slope has zero rise, and so is parallel to the x axis. A line with zero run has **no slope** (cannot divide by zero) and is parallel to the y axis.

Two lines are **perpendicular** (meet at right angles) if the product of their slopes is -1.

Example The slope of the line passing through the points A (1, 6) and B (0 , -2) is

$$\frac{\text{Rise}}{\text{Run}} = \frac{6-(-2)}{1-0} = \frac{8}{1} = 8 .$$

which is a very steep line (it rises quickly in the direction of the y axis over a short run along the x axis, i.e. 8 units up for every 1 unit along).

Do either of the points (2, 9) and (-1, -10) lie on this line AB ?

For any point to be on the line, it must make the same slope of 8 with both points (1, 6) and (0, -2).
For (2, 9) the slope of the line segment connecting this point to (1, 6) is $\frac{\text{Rise}}{\text{Run}} = \frac{6-9}{1-2} = \frac{-3}{-1} = 3$
which is not the required slope of 8, and so (2, 9) is not a point on AB.

For (-1, -10) the slope of the line segment connecting this point to (1, 6) is $\frac{\text{Rise}}{\text{Run}} = \frac{6-(-10)}{1-(-1)} = \frac{16}{2} = 8$
which is the required slope and so (-1, -10) is a point on AB.

It is easy to check that slope of the line segment connecting (-1,-10) to (0, -2) is also 8. This must occur if they both connect to the point (1, 6) with the same slope, i.e. they must be **collinear** (lying on the same line).

Example The slope of a line passing through the point (4, 2) is -½ . Find two other points on this line.

As the slope is -½ , this means that the ratio of the rise to the run for this line is -1 in the y direction for each 2 units in the x direction. So beginning at the point (4, 2) we can find another point on the line by decreasing the y coordinate by 1 and increasing the x coordinate by 2, to give (4 + 2, 2 - 1) = (6, 1). Repeating this process would give yet another point on the line, (6 + 2, 1 - 1) = (8, 0).

We can put these values into a table, follow the pattern to fill in other points, and try to identify the relationship between the x and y coordinates for points on this line.

x coordinate	y coordinate	Ordered pair
0	4	(0, 4)
2	3	(2, 3)
4	2	(4, 2)
6	1	(6, 1)
8	0	(8, 0)
10	-1	(10, -1)

For each increase in x by 2, we reduce the value of y by 1. This relationship is expressed in the equation

$$y = -\tfrac{1}{2}x + 4$$

The slope of -½ is easily seen in this equation, as is the y coordinate where the line intersects the y axis, namely 4.

Every point on the line must satisfy this equation, and any point that satisfies this equation must lie on this line. So just as we have seen with circles, there is a defining algebraic equation for a line graph in the rectangular coordinate system.

Equations of Lines

The set of ordered pairs whose x and y coordinates satisfy an equation of the form

y = mx + b for some numbers **b** and **m**,

forms a graph in the shape of a line. The slope of the line is the value **m** and the point where the line intersects the y axis is (0, **b**).

Any linear graph is a set of ordered pairs with coordinates related by an equation of this form **y = mx + b** . It is called the **standard form** of the equation of a line.

The **x intercept** of a line is the point (with 0 as y coordinate) at which the line intersects the x axis.
The **y intercept** of a line is the point (with 0 as x coordinate) at which the line intersects the y axis.

Example Find three points on each of the following lines and then plot their graphs.
(a) y = x + 2 (b) y = -x + 6 (c) 2y = 2x – 1

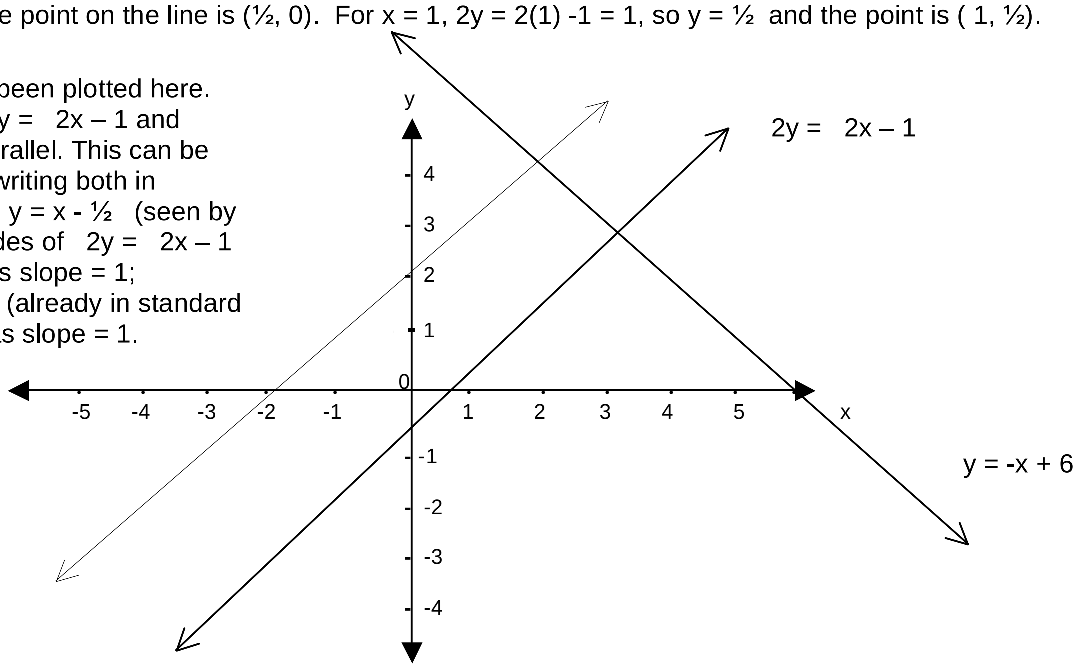
To find points on a given line, choose a value for x and determine the corresponding value for y from the equation of the line. Plot the points and join them to form the given line.

For y = x + 2, let x = 0 and the corresponding y = 0 + 2 = 2. The point on the line is (0, 2).
For another point choose another x, say x = 1, then y = 1 + 2 = 3. The point on the line is (1, 3).
For another point choose another x, say x = -1, then y = -1 + 2 = 1. The point on the line is (-1, 1).

For y = -x + 6, if x = 0, the point on the line is (0, 6). For x = 6, the point on the line is (6, 0).
For x = 3 the point on the line is (3, 3) .

For 2y = 2x – 1, if x = 0 then 2y = 0 -1, so y = -½ and the point is (0, -½). For x = ½, 2y = 2(½) -1 = 0 so y = 0 and the point on the line is (½, 0). For x = 1, 2y = 2(1) -1 = 1, so y = ½ and the point is (1, ½).

The lines have been plotted here. Clearly lines 2y = 2x – 1 and y = x + 2 are parallel. This can be easily seen by writing both in standard form: y = x - ½ (seen by dividing both sides of 2y = 2x – 1 by 2) and so has slope = 1; and y = x + 2 (already in standard form) and so has slope = 1.



Recall that the slope of each line is the number next to the x term in each equation. So we see that both lines have the same slope and so that is why their graphs are parallel.

The slope of the third line is -1 since y = -x + 6 is in standard form with -1 next to the x term. Since the product of -1 and 1 (the slope of the other two lines) equals -1, we know that y = -x + 6 is perpendicular to the lines 2y = 2x – 1 and y = x + 2, as shown in the graph.

Example What is the equation of the line passing through the points (2, 2) and (0, -5)?

The slope of this line is $\frac{\text{Rise}}{\text{Run}} = \frac{2-(-5)}{2-0} = \frac{7}{2} = 3.5$ and its y intercept is given, namely (0, -5).
So in standard form its equation would be y = 3.5x - 5 or in the usual fractional form $y = {}^{7}/_{2}\,x - 5$.
Doubling both sides gives another form of the equation, 2y = 7x -10.
These different forms all represent the same line graph through the points (2, 2) and (0, -5).

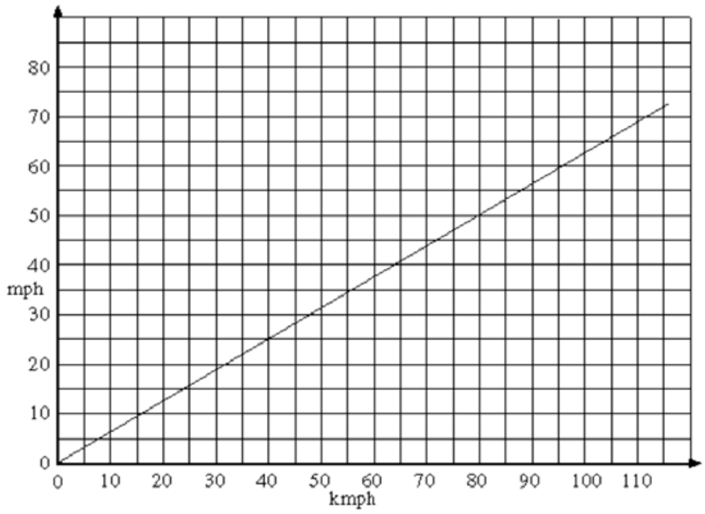
Practice Problems – Lines and their Slopes

1. Find the equation of the line which passes through
(a) the origin and has slope 3
(b) the points (0, 1) and (-2, 3)
(c) the point (-4, 6) and has slope of ½
(d) the point (2, 7) and is parallel to the x axis.

2. Determine the slopes of the following lines by writing them in standard form.
Which of the lines are parallel or perpendicular?

(i) $y = \frac{4}{3}x + 11$ (ii) $y = \frac{3}{4}x - 20$ (iii) $3y = 4x + 6$
(iv) $3y - 4x = 0$ (v) $3x - 4y + 2 = 0$ (vi) $y = 8 - \frac{3}{4}x$

3. The vertical axis in this graph shows miles per hour and the horizontal axis shows kilometres per hour. The line on the graph can be used to convert speeds between mph and kmph.



(a) Use the graph to estimate the speeds
in miles per hour for (i) 100 kmph
(ii) 80 kmph (iii) 40 kmph
in kilometres per hour for (iv) 30 mph
(v) 70 mph (vi) 10 mph

(b) Use two points to find the slope of this linear graph.

(c) What is its y intercept?

(d) Write the equation for this line in standard form.

4. The rule for converting between degrees Celsius and degrees Fahrenheit is

$y = \frac{9}{5}x + 32$ where y is the degrees Fahrenheit and x is the degrees Celsius.

(a) What would the degrees Fahrenheit be for (i) 0 °C (ii) 15 °C (iii) 30 °C?

(b) Plot these points on coordinates axes and graph the line $y = \frac{9}{5}x + 32$. Use your graph to find the degrees Celsius corresponding to (i) 104 °F (ii) 14 °F

(c) Find the slope and x and y intercepts for this graph.

5. A cylindrical water tank holding 360 litres develops a leak with water draining out of a small hole at a steady 30 litres per hour.

(a) How much water would be in the tank after (i) 1 hour (ii) 2 hours (iii) 3 hours?

(b) Draw a graph to show how the level of water changes over time based on these points. What is the equation for this graph showing the linear relationship between water left in the tank and time?

(c) After 8 hours the tank stops leaking as the level of water has reached a point below the hole. It takes one hour to repair and then the tank is refilled at a rate of 60 litres per hour. Draw a new graph to show how the level of water changes over time in this scenario.

(d) How does the slope of the graph change over time?

6. The number of home-schooled children in a particular country (in thousands) for selected years is shown in the table.

School Year	x	y = no. of Children
1993	3	588
1994	4	735
1995	5	800
1996	6	920
1997	7	1100

(a) Graph the points (x, y) to see the trend in the number of home-schooled children from 1993 to 1997.

(b) What was the average rate of increase (slope) between
(i) 1993 and 1994 (ii) 1994 and 1996 (iii) 1993 and 1997?

(c) Draw in the line connecting the point with x = 3 and the point with x = 7.

(d) Estimate the number of children home-schooled in 1990 (x = 0) from this line.

(e) Does this line accurately predict the value of the number of home schooled children in 1995? Why?

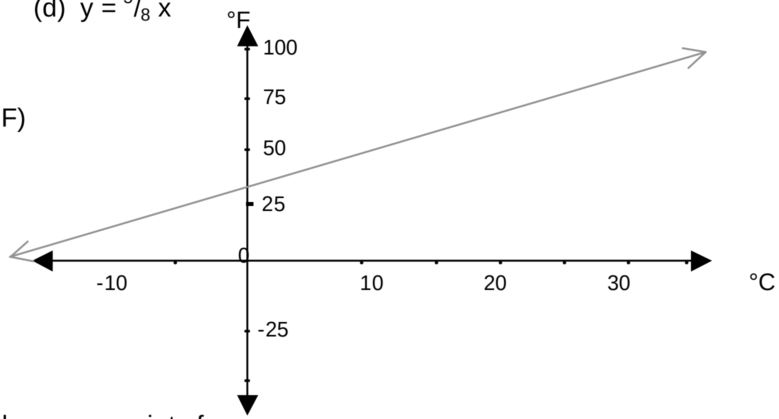
Solutions to Practice Problems - Lines and their Slopes

1. (a) y = 3x (b) slope = -1, y intercept = (0, 1) so y = -x + 1 (c) Increase y by 1 for each increase in x by 2, so get other points are (-2, 7) and y intercept (0, 8), equation is y = ½ x + 8
(d) slope = 0 so equation is just y = 7.

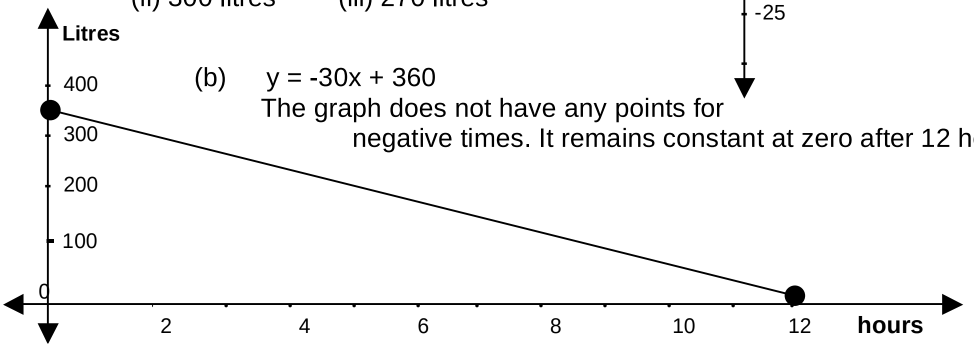
2. Slopes are (i) $^{4}/_{3}$ (ii) $^{3}/_{4}$ (iii) $^{4}/_{3}$ (iv) $^{4}/_{3}$ (v) $^{3}/_{4}$ (vi) $-^{3}/_{4}$
So (i), (iii) and (iv) parallel; (ii) and (v) parallel; and (vi) perpendicular to (i), (iii) and (iv).

3. (a) (i) 62.5 mph (ii) 50 mph (iii) 25 mph (iv) 48 kmph (v) 112 kmph (vi) 16 kmph
(b) Slope = $^{5}/_{8}$ or 0.625 (c) (0, 0) (d) y = $^{5}/_{8}$ x

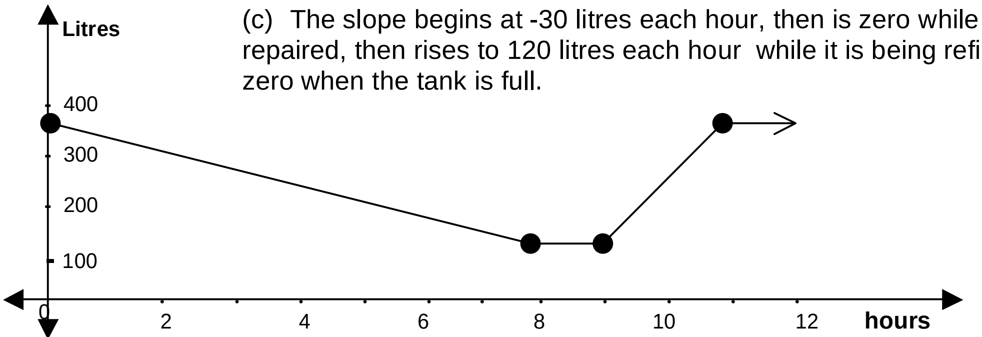
4.(a) (i) 32°F (ii) 59°F (iii) 86°F
(b) (i) 40°C (ii) -10°C
(c) Slope $^{9}/_{5}$, x intercept (-17.8°C, 0°F)
and y intercept (0°C, 32°F)



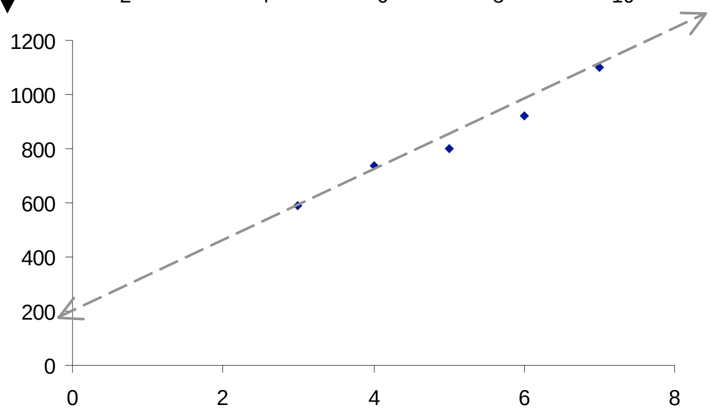
5. (a) (i) 330 litres
(ii) 300 litres (iii) 270 litres



(b) y = -30x + 360
The graph does not have any points for negative times. It remains constant at zero after 12 hours.



(c) The slope begins at -30 litres each hour, then is zero while the tank is being repaired, then rises to 120 litres each hour while it is being refilled, then goes back to zero when the tank is full.



6. (a) See plotted points.

(All answers are in thousands of children)
(b) (i) 147 per year
(ii) 92.5 per year
(iii) 128 per year

(c) see dotted line on graph.

(d) About 200 (exact 204)

(e) No. As the line lies above the data point of y = 800 for x = 5, its estimate is too high. The line estimates about y = 844 for x = 5.

Diagrams for Exercises 1.6 and 2.3 taken from
http://www.cimt.plymouth.ac.uk/projects/mepres/book7/bk7i3/bk7_3i6.htm June 2007